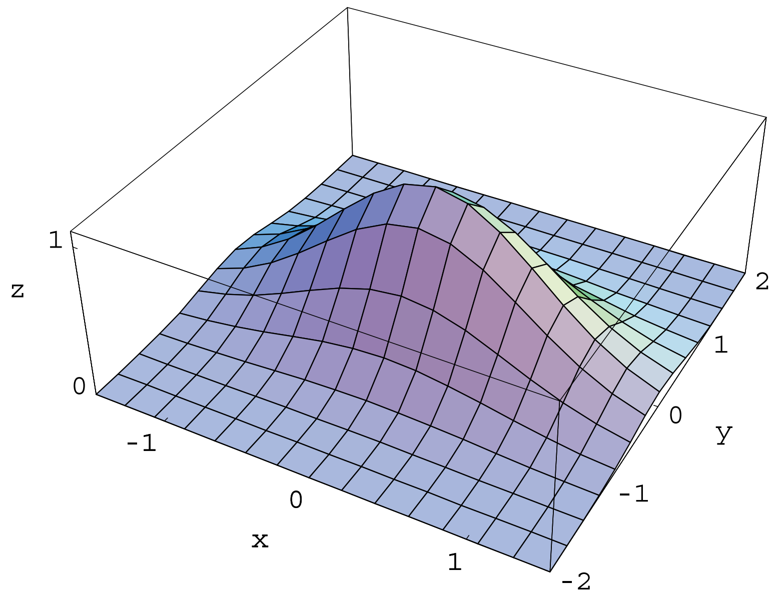
Double integrals

If you have a function of two variables, $f(x,y)$, the graph of the function gives you a surface, which might look like this:



The two horizontal axes are the *x*- and *y*-axes, and the z axis goes vertically upwards.

We can extend the concept of an integral to calculate the volume of the solid region under a part of the surface.

In order to calculate this volume we need to evaluate a double integral of the function over the region *R*. This is written as $\iint_R f(x,y)\,dx\,dy$.

The mathematical definition of a double integral is just an extension of the definition for an ordinary integral. It involves taking the limit of the sum of the volumes of a large number of rectangular blocks.

In practice you work out a double integral by first integrating with respect to one of the variables (*x* or *y*) then integrating with respect to the other variable.

The limits of integration must be chosen so that they describe the boundaries of the region *R*. This is the footprint of the volume on the *x*-*y* plane.

To evaluate these integrals, we treat x as a constant when integrating with respect to y, and y as a constant when integrating with respect to x.

The case where the region *R* is a rectangle parallel to the axes is straightforward.

Question 1.1

Evaluate the double integral $\int\limits_{x=10}^{15} \int\limits_{y=0}^{5} \frac{2x+y}{3000}\, dy\, dx$.

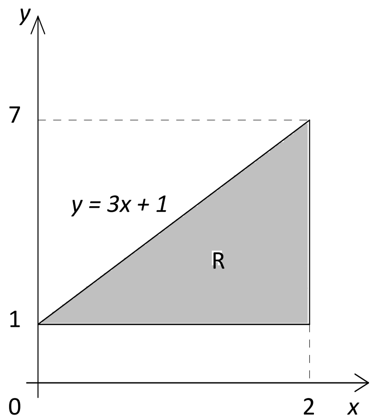
However, in other cases the limits will be a function of x when first integrating with respect to y, and of y when first integrating with respect to x. If you need to work out the limits, you will need to consider the required region, as illustrated in the following example.

Example

Integrate xy^2 over the region R shown below:

(i) by first integrating with respect to *y* and then integrating with respect to *x*

(ii) by first integrating with respect to *x* and then integrating with respect to *y.*



Solution

(i) In this part *x* is the variable in the 'outside' integral and the range of *x* values is $0 \le x \le 2$. For any given value of *x* the range of values of *y* is $1 \le y \le 3x+1$. This gives us our limits of integration.

$$I = \int_0^2 \left\{ \int_1^{3x+1} xy^2 \; dy \right\} dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} \right]_1^{3x+1} dx$$

$$= \int_0^2 \frac{x(3x+1)^3}{3} - \frac{x}{3} \; dx$$

$$= \int_0^2 9x^4 + 9x^3 + 3x^2 dx$$

$$= \left[\frac{9x^5}{5} + \frac{9x^4}{4} + x^3 \right]_0^2$$

$$= 101.6$$

(ii) In this part y is the variable in the 'outside' integral and the range of y values is $1 \le y \le 7$. For any given value of y the range of values of x is $\frac{y-1}{3} \le x \le 2$. This gives us our limits of integration.

$$I = \int_1^7 \left\{ \int_{\frac{y-1}{3}}^{2} xy^2 \, dx \right\} dy$$

$$= \int_1^7 \left[\frac{x^2 y^2}{2} \right]_{\frac{y-1}{3}}^{2} dy$$

$$= \int_1^7 2y^2 - \frac{(y-1)^2 y^2}{18} \, dy$$

$$= \left[\frac{2y^3}{3} - \frac{1}{18}\left(\frac{y^5}{5} - \frac{2y^4}{4} + \frac{y^3}{3} \right) \right]_1^7$$

$$= 102.26 - 0.66 = 101.6$$

The following results can be used.

$$\iint_R af(x,y) + bg(x,y) \, dx \, dy = a\iint_R f(x,y) \, dx \, dy + b\iint_R g(x,y) \, dx \, dy$$

$$\iint_R f(x,y) \, dx \, dy = \iint_{R_1} f(x,y) \, dx \, dy + \iint_{R_2} f(x,y) \, dx \, dy$$

Here R is the combined region consisting of the separate regions R_1 and R_2, *ie* $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \varnothing$.

Question 1.2

Evaluate the double integral $\int_0^3 \int_y^3 xe^{2x+3y} \, dx \, dy$.

Sometimes double integrals with explicit limits are written with the dx and dy at the front next to the integral sign each belongs to. In this case the last question would be written as $\int_0^3 dy \int_y^2 dx\, xe^{2x+3y}$.

The results for double integrals can be extended to triple integrals.

Question 1.3

Evaluate $\int_1^3 \int_0^{3z} \int_1^{2+z} e^{x+y}\, dx\, dy\, dz$.

If the integrand (the expression being integrated) factorises into functions of the three separate variables, and the limits are constants, you can multiply together the individual integrals. This is not obvious but it is true. This would mean:

$$\int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy\, f(x)g(y) = \int_{x_1}^{x_2} f(x)dx \int_{y_1}^{y_2} g(y)dy$$

Question 1.4

Evaluate $\int_0^1 dx \int_0^2 dy \int_0^3 dz\, xyz$.

Swapping the order of integration

We can change the order of integration.

Example

Given that $0<t<s<5$, calculate the value of $\int_{s=0}^{5} s^2 \int_{t=0}^{s} t\, dt\, ds$, first by using this order and then by swapping the order of the two variables.

Solution

Firstly, we will use the order given:

$$\int_{s=0}^{5} s^2 \int_{t=0}^{s} t\, dt\, ds = \int_{s=0}^{5} s^2 \left[\frac{t^2}{2}\right]_0^s ds = \int_{s=0}^{5} \frac{s^4}{2} ds = \left[\frac{s^5}{10}\right]_0^5 = 312.5$$

Previously we integrated from $t=0$ to $t=s$, then from $s=0$ to $s=5$. Reversing this, we integrate from $s=t$ to $s=5$ then from $t=0$ to $t=5$. To get the limits for s, see what it is bounded by in the inequality $0<t<s<5$. So:

$$\int_{s=0}^{5} s^2 \int_{t=0}^{s} t\, dt\, ds = \int_{t=0}^{5} t \int_{s=t}^{5} s^2\, ds\, dt$$

Carrying out this integration:

$$\int_{t=0}^{5} t \int_{s=t}^{5} s^2\, ds\, dt = \int_{t=0}^{5} t \left[\frac{s^3}{3}\right]_t^5 dt = \int_{t=0}^{5} t\, \frac{125-t^3}{3}\, dt = \frac{1}{3}\left[\frac{125t^2}{2} - \frac{t^5}{5}\right]_0^5 = 312.5$$

Question 1.5

(i) Carry out the integration $\int_{y=0}^{30} e^{-\mu y} \mu \int_{x=0}^{y} e^{-\nu x} e^{-\delta x} \, dx \, dy$, where $0 < x < y < 30$.

(ii) Confirm that you get the same answer if you reverse the order of integration and use the values $\nu = 0.001$, $\delta = 0.06$, $\mu = 0.005$.

Solutions

Solution 1.1

$$\int_{x=10}^{15}\int_{y=0}^{5}\frac{2x+y}{3000}\,dy\,dx=\frac{1}{3000}\int_{10}^{15}\left[2xy+\tfrac{1}{2}y^2\right]_0^5 dx$$

$$=\frac{1}{3000}\int_{10}^{15}10x+12.5\,dx$$

$$=\frac{1}{3000}\left[5x^2+12.5x\right]_{10}^{15}$$

$$=\frac{11}{48}$$

Solution 1.2

We need to integrate by parts.

Use $u=x$, and $\dfrac{dv}{dx}=e^{2x+3y}$ so that $\dfrac{du}{dx}=1$ and $v=\dfrac{1}{2}e^{2x+3y}$, to give:

$$\int_0^3\int_y^3 xe^{2x+3y}\,dx\,dy=\int_0^3\left\{\left[\frac{1}{2}xe^{2x+3y}\right]_y^3-\int_y^3\frac{1}{2}e^{2x+3y}dx\right\}dy$$

$$=\int_0^3\frac{3}{2}e^{3y+6}-\frac{1}{2}ye^{5y}-\left[\frac{1}{4}e^{2x+3y}\right]_y^3 dy$$

$$=\int_0^3\frac{3}{2}e^{3y+6}-\frac{1}{4}e^{3y+6}+\left(\frac{1}{4}-\frac{1}{2}y\right)e^{5y}\,dy$$

$$=\int_0^3\frac{5}{4}e^{3y+4}+\left(\frac{1}{4}-\frac{1}{2}y\right)e^{5y}\,dy$$

The first part is easy to integrate but the second term requires integration by parts, this time with $u=\frac{1}{4}-\frac{1}{2}y$ and $\frac{dv}{dx}=e^{5y}$. This gives:

$$\left[\frac{5}{12}e^{3y+6}\right]_0^3+\left[\left(\frac{1}{4}-\frac{1}{2}y\right)\frac{e^{5y}}{5}\right]_0^3+\frac{1}{10}\int_0^3 e^{5y}dy$$

$$=\left[\frac{5}{12}e^{3y+6}+\frac{1}{5}\left(\frac{1}{4}-\frac{1}{2}y\right)e^{5y}+\frac{1}{50}e^{5y}\right]_0^3$$

$$=\left(\frac{5}{12}e^{15}-\frac{1}{4}e^{15}+\frac{1}{50}e^{15}\right)-\left(\frac{5}{12}e^{6}+\frac{1}{20}+\frac{1}{50}\right)$$

$$=\frac{14}{75}e^{15}-\frac{5}{12}e^{6}-\frac{7}{100}$$

$$=610,048$$

Solution 1.3

$$\int_1^3\int_0^{3z}\int_1^{2+z}e^{x+y}\,dx\,dy\,dz=\int_1^3\int_0^{3z}\left[e^{x+y}\right]_1^{2+z}dy\,dz$$

$$=\int_1^3\int_0^{3z}e^{2+z+y}-e^{1+y}dy\,dz$$

$$=\int_1^3\left[e^{2+z+y}-e^{1+y}\right]_0^{3z}dz$$

$$=\int_1^3 e^{2+4z}-e^{1+3z}-e^{2+z}+e\,dz$$

$$=\left[\tfrac{1}{4}e^{2+4z}-\tfrac{1}{3}e^{1+3z}-e^{2+z}+ez\right]_1^3$$

$$=\tfrac{1}{4}e^{14}-\tfrac{1}{3}e^{10}-\tfrac{1}{4}e^{6}-e^{5}+\tfrac{1}{3}e^{4}+e^{3}+2e$$

$$=293,103$$

Solution 1.4

$$\int_0^1 dx\int_0^2 dy\int_0^3 dz\,xyz=\int_0^1 x\,dx\times\int_0^2 y\,dy\times\int_0^3 z\,dz=\frac{1}{2}\times2\times\frac{9}{2}=4\frac{1}{2}$$

Solution 1.5

(i) Integrating first with respect to x:

$$\int_{y=0}^{30} e^{-\mu y}\mu \int_{x=0}^{y} e^{-\nu x}e^{-\delta x}\,dx\,dy$$

$$=\int_{y=0}^{30} e^{-\mu y}\mu\left[-\frac{1}{\nu+\delta}e^{-(\nu+\delta)x}\right]_0^y\,dy$$

$$=\int_{y=0}^{30} e^{-\mu y}\mu\left[-\frac{1}{\nu+\delta}e^{-(\nu+\delta)x}\right]_0^y\,dy$$

$$=\int_{y=0}^{30} e^{-\mu y}\mu\left(\frac{1}{\nu+\delta}(1-e^{-(\nu+\delta)y})\right)dy$$

Then integrating with respect to y, we get:

$$\frac{\mu}{\nu+\delta}\int_{y=0}^{30} e^{-\mu y}-e^{-(\nu+\delta+\mu)y}\,dy$$

$$=\frac{\mu}{\nu+\delta}\left[-\frac{1}{\mu}e^{-\mu y}+\frac{1}{\nu+\delta+\mu}e^{-(\nu+\delta+\mu)y}\right]_0^{30}$$

$$=\frac{\mu}{\nu+\delta}\left[\frac{1}{\mu}-\frac{1}{\mu}e^{-30\mu}+\frac{1}{\nu+\delta+\mu}e^{-30(\nu+\delta+\mu)}-\frac{1}{\nu+\delta+\mu}\right]$$

(ii) Evaluating (i) numerically, we get 1.213.

Changing the order of integration, consider $0<x<y<30$. So:

$$\int_{y=0}^{30} e^{-\mu y}\mu \int_{x=0}^{y} e^{-\nu x}e^{-\delta x}\,dx\,dy=\int_{x=0}^{30} e^{-\nu x}e^{-\delta x}\int_{y=x}^{30} e^{-\mu y}\mu\,dy\,dx$$

Then integrating with respect to y, we get:

$$\int_{x=0}^{30} e^{-\nu x} e^{-\delta x} \left[-e^{-\mu y} \right]_x^{30} dx$$

$$= \int_{x=0}^{30} e^{-\nu x} e^{-\delta x} \left[e^{-\mu x} - e^{-30\mu} \right] dx$$

$$= \int_{x=0}^{30} e^{-(\nu+\delta+\mu)x} - e^{-(\nu+\delta)x} e^{-30\mu} \; dx$$

$$= \left[-\frac{1}{\nu+\delta+\mu} e^{-(\nu+\delta+\mu)x} + \frac{1}{\nu+\delta} e^{-(\nu+\delta)x} e^{-30\mu} \right]_0^{30}$$

$$= \frac{1}{\nu+\delta+\mu}(1 - e^{-30(\nu+\delta+\mu)}) + \frac{1}{\nu+\delta}(e^{-30(\nu+\delta+\mu)} - e^{-30\mu})$$

Substituting in the given values, we also get 1.213, confirming that the two methods give the same numerical value.

